

LARGE RAINBOW MATCHINGS IN GENERAL GRAPHS

RON AHARONI, ELI BERGER, MARIA CHUDNOVSKY, DAVID HOWARD, AND PAUL SEYMOUR

ABSTRACT. By a theorem of Drisko, any $2n - 1$ matchings of size n in a bipartite graph have a partial rainbow matching of size n . Barát, Gyárfás and Sárközy conjectured that if n is odd then the same is true also in general graphs, and that if n is even then $2n$ matchings of size n suffice. We prove that any $3n - 2$ matchings of size n have a partial rainbow matching of size n .

1. INTRODUCTION

Given a system $\mathcal{C} = (C_1, \dots, C_m)$ of sets of edges in a graph, a *partial rainbow matching* for \mathcal{C} is a matching consisting of a choice of one edge from C_i , $i \in S$ for some $S \subseteq [m]$. In this paper we shall mainly consider the case in which the sets C_i are themselves matchings, and we shall be interested in questions of the form “how many matchings of size k are needed to guarantee the existence of a partial rainbow matching of size m .” It is conjectured [2] that n matchings of size $n + 1$ in a bipartite graph have a rainbow matching of size n , and there is no known example refuting the possibility that n matchings of size $n + 2$ in a general graph have a rainbow matching of size n . It is known that in the bipartite case n matchings of size $\lceil \frac{3}{2}n \rceil$ [4] or size $n + o(n)$ [10] suffice for the existence of a rainbow matching of size n .

A surprising jump occurs when we insist that the matchings are of size n : we need to take $2n - 1$ such matchings to guarantee a rainbow matching of size n . The following example shows that $2n - 1$ is best possible:

Example 1.1. Let M_i , $1 \leq i \leq n-1$ to be all equal to one of the two perfect matchings in C_{2n} and M_i , $i \leq 2n-2$ to be all equal to the other perfect matching. This is a system of $2n - 2$ matchings of size n that does not have a rainbow matching of size n .

The fact that $2n - 1$ matchings suffice is essentially due to Drisko [7], who proved the following special case:

Theorem 1.2. *Let A be an $m \times n$ matrix in which the entries of each row are all distinct. If $m \geq 2n - 1$, then A has a transversal, namely a set of n distinct entries with no two in the same row or column.*

In [2] this was formulated in the rainbow matchings setting, and given a short proof.

Theorem 1.3. *Any family $\mathcal{M} = (M_1, \dots, M_{2n-1})$ of matchings of size n in a bipartite graph possesses a rainbow matching.*

In [4] it was shown that Example 1.1 is the only instance in which $2n - 2$ matchings do not suffice. In [5] Theorem 1.3 was strengthened, using topological methods:

Theorem 1.4. *If M_i , $i = 1, 2n - 1$ are matchings in a bipartite graphs satisfying $|M_i| = \min(i, n)$ for all $i \leq 2n - 1$ then there exists a rainbow matching of size n .*

Barát, Gyárfás and Sárközy considered the same problem in general graphs, and suggested the following:

The research of the first author was supported by BSF grant no. 2006099, by the Technion’s research promotion fund, and by the Discont Bank chair.

The research of the second author was supported by BSF grant no. 2006099 and by ISF grant no.

The research of the third author was supported by BSF grant no. 2006099, and NSF grants DMS-1001091 and IIS-1117631.

The research of the fourth author was supported by BSF grant no. 2006099, and by ISF grants Nos. 779/08, 859/08 and 938/06.

The research of the fifth author was supported by .

Conjecture 1.5. [6] *For n even any $2n$ matchings of size n have a partial rainbow matching of size n , and for n odd any $2n - 1$ matchings have a rainbow matching of size n .*

Example 1.6. The following example shows that for n even $2n - 1$ matchings of size n in a graph do not necessarily have a rainbow matching of size n . Let $n = 2k$. Number the vertices of C_{2n} as v_1, v_2, \dots, v_{4k} , and let K be the matching $\{v_1v_3, v_2v_4, v_5v_7, v_6v_8, \dots, v_{4n-3}v_{4n-1}, v_{4n-2}v_{4k}\}$. Let $M_0 = K$, let \mathcal{M} be the family consisting of K and of $n - 1$ copies of each of the two matchings of size n in C_{2n} . Then \mathcal{M} does not have a rainbow matching of size n . If there was, it would have to contain an edge from K , and without loss of generality this edge is v_1v_3 . But then no edge can be chosen from any other matching in \mathcal{M} that contains the vertex v_2 .

Question: is this the only example?

The aim of this paper is to prove:

Theorem 1.7. *$3n - 2$ matchings of size n in any graph have a partial rainbow matching of size n .*

2. PRELIMINARIES AND NOTATION

We shall use the following paths-related notation. The first vertex on a path P is denoted by $in(P)$, and its last vertex by $ter(P)$. The edge set of P is denoted by $E(P)$, and its vertex set by $V(P)$. Given a family of paths \mathcal{P} , we write $E[\mathcal{P}] = \bigcup_{P \in \mathcal{P}} E(P)$. For a path P and a vertex v on it, we denote by Pv the part of P up to and including v , and by vP the part from v (including v) and on. If P, Q are paths such that $in(Q) = ter(P)$ we write PQ for the trail (namely a path that is not necessarily simple) resulting from the concatenation of P and Q . We denote by \overleftarrow{P} the path P traversed in the opposite direction to that of P .

Let F be a matching in a graph, and let K be a set of edges disjoint from F . A path P is said to be $K - F$ -alternating if every odd-numbered edge of P belongs to K and every even-numbered edge belongs to F . If there is no restriction on the odd edges of P then we just say that it is F -alternating. If both $in(P)$ and $ter(P)$ do not belong to $\bigcup F$ then P is said to be *augmenting*. The origin of the name is that in such a case $E(P) \triangle F$ is a matching larger than F . Famously, the converse is also true:

Lemma 2.1. *If F, G are matchings and $|G| > |F|$ then $E(F) \cup E(G)$ contains an F -alternating augmenting path.*

Proof. Viewed as a multigraph, the connected components of $E(F) \cup E(G)$ are cycles (possibly digons) and paths that alternate between G and F edges. Since $|G| > |F|$ one of these paths contains more edges from G than from F , and is thus F -augmenting. \square

Definition 2.2. Let F be a matching, let K be a set of edges disjoint from F , and let a be any vertex. A vertex $v \in \bigcup M$ is said to be *oddly K -reachable* (resp. *evenly K -reachable*) from a if there exists an odd (respectively even) $K - F$ -alternating path starting with an edge $ab \in K$ and ending at v . Note that being an odd alternating path means ending with an edge from K , and being an even alternating path means ending with an edge of F . Let $OR(a, K, F)$ be the set of vertices oddly reachable from a , $ER(a, K, F)$ the set of vertices evenly reachable from a (since the single vertex path is considered to be even, we have $a \in ER(a, K, F)$), and let $DR(a, K, F) = (OR(a, K, F) \cap ER(a, K, F)) \cup \{a\}$. We say that v is *oddly K -reachable* (respectively *evenly K -reachable*) if it is oddly (respectively evenly) reachable from some vertex not belonging to $\bigcup F$.

Note that there exists a $K - F$ augmenting alternating path if and only if $OR(K, F) \not\subseteq \bigcup F$.

Definition 2.3. A graph G is called *hypomatchable* if $G - v$ has a perfect matching for every $v \in V(G)$.

Lemma 2.4. *Let F be a matching in a graph G , let $K = E \setminus F$, and suppose that $V(G) \setminus \bigcup F$ consists of a single vertex a . Then a vertex x belongs to $ER(a, K, F)$ if and only if $G - x$ has a perfect matching.*

Proof. Suppose that there exists a matching M of $G - x$. Then the $F - M$ -alternating path starting at x with an edge of F must terminate at a with an edge of M , which proves that $x \in OR(a, K, F)$. If $x \in OR(a, K, F)$

then taking L to be the odd $a - x$ F -alternating path reaching x and letting $M = F \triangle L$ yields a perfect matching of $G - x$.

□

Note that $x \in OR(a, K, F)$ if and only if $F(x) \in ER(a, K, F)$. Hence the lemma implies:

Corollary 2.5. *Let F be a matching in a graph G , let $K = E(G) \setminus F$, and let a be the single vertex in $V(G) \setminus \bigcup F$. Then G is hypomatchable if and only if $V(G) = DR(a, K, F)$.*

3. SNICK-BERRY SWITCHES

Let G be a graph, let F be a matching in it, and write $K = E \setminus F$. The pair (G, F) is called a *snick-berry tree* if it can be obtained from a rooted tree T with root r as follows. Subdivide every edge $e = st$ of T , where s is the vertex nearer to r , by a vertex $m(e)$. Replace each vertex s of the original tree by a hypomatchable graph $H(s)$, such that $F \upharpoonright H(s)$ matches all vertices apart from a single vertex $r(s)$. For every child t of s choose some vertex $v \in H(s)$ and connect it to $m(st)$ by an edge of K . Connect $m(st)$ to $r(t)$ by an edge of F . The sets $V_t = V(H(t))$ are called *islands*. We say that T *guides* the snick-berry tree.

A pair (G, F) of a graph G and a matching F in it is called a *snick-berry switch*, or SBS for short, if each of its connected components is of one of two types: a snick-berry tree, or a component on which F induces a perfect matching. (This object was first studied in [8], where some illustrations of it can be found.)

Theorem 3.1. *Let $G = (V, E)$ be a graph, let F be a matching in G , and let $K = E \setminus F$. Suppose that:*

- (1) F is a matching of maximal size in G , and
- (2) For every $L \subsetneq K$ we have $OR(L, F) \subsetneq OR(K, F)$.

Then the pair (G, F) is an SBS.

Proof. It suffices to show that if G satisfies the conditions of the theorem and is connected, then it is a snick-berry tree. If G consists of a single edge belonging to F then the lemma is true, with the tree consisting of a single vertex. So, we may assume that this is not the case.

We construct the tree T guiding the snick-berry tree inductively, by adding at the i -th stage an island V_{t_i} with a hypomatchable graph $H(t_i)$. We call the tree obtained after adding the i -th island T_i . The inductive assumption will be that for each island V_t in T_i there holds:

- (1) $V_t = DR(r(t), K, F)$, and
- (2) $r(t) \in OR(r(T_i), K, F) \setminus ER(r(T_i), K, F)$.

If $OR(K, F) = \emptyset$ then by condition (2) $K = \emptyset$, meaning that F is a perfect matching in G (actually, with the assumption of connectedness, a single edge), and the theorem is true. So, we may assume that $OR(K, F) \neq \emptyset$. This means that there exists a vertex $a \notin \bigcup F$. Define t_1 as r , the root of T , and let T_1 be the tree consisting of the single vertex t_1 . Let $r(t_1) = a$, and let $V_{t_1} = DR(a, K, F)$.

Suppose that T_i has been defined. If $\bigcup \{V_t \mid t \in V(T_i)\} \cup \{m(st) \mid st \in E(T_i)\} = V$ then we halt the construction and let $T = T_i$. Otherwise, we choose an edge xy where $x \in V_s$, $s \in V(T_i)$ and $y \notin \bigcup \{V_t \mid t \in V(T_i)\} \cup \{m(st) \mid st \in E(T_i)\} = V$. By its choice, $y \in OR(r(s), K, F)$ and since by the induction hypothesis $V_s = DR(r(s), K, F)$ and $r(s) \in OR(a, K, F)$, we have $y \in OR(a, K, F)$. By the inductive assumption $y \notin DR(r(s), K, F)$, meaning that $y \notin ER(a, K, F)$, proving (b) for T_{i+1} .

By condition (1), $y \in \bigcup F$. Let z be the vertex connected by F to y , construct T_{i+1} by adding a descendant t_{i+1} of s to T_i , and let $V_{t_{i+1}} = DR(z, K, F)$. Let $z = r(t_{i+1})$. By (2) above, there is no other edge, except for xy , that connects y with any V_t , $t \in V(T_i)$. Also, there is no edge connecting V_r to $V_{t_{i+1}}$, since such an edge would generate a $K - F$ alternating path showing that $y \in DR(a, K, F)$, implying that $y \in V_r$, contrary to the choice of y .

By the construction, for every $t \in V(T)$ and every vertex $v \in V_t$ there exists an even $K - F$ alternating path $EP(v)$ from a to v going only through islands V_s , for s belonging to the path in T from r to t , and the

bridges between them. Also, for every $v \in V_t$ there exists an even $K - F$ -alternating path $EQ(v)$ from $r(t)$ to v .

Finally, we have to show that if an edge $uv \in E(G)$ satisfies $u \in V_s$ and $v \notin V_s$ then either

- (1) $v = m(st)$ for a direct descendant t of s , or
- (2) $u = r(s)$ and $v = m(ps)$ for the father p of s in the tree T .

Suppose, to the contrary, that there exists an edge uv contradicting this assertion. There are two cases to consider:

- $v \in V_t$ for some $t \in V(T)$. Let p be the father of t . Then $EP(u)$ concatenated with $\overleftarrow{EQ(v)}$ shows that $m(pt) \in ER(a, K, F)$, contrary to (b) above.
- $v = m(pq)$ where p is the father of the vertex q of T . This cannot happen, because the deletion of uv does not curtail $OR(a, K, F)$, contrary to assumption (2) in the theorem.

□

Remark 3.2. From the proof it follows that if there exists an edge joining a vertex in V_s and V_t where s is not a descendant of t then $r(t) \in OR(K \cup \{e\}, F) \setminus OR(K, F)$.

4. MULTICOLORED ALTERNATING PATHS AND PROOF OF THEOREM 1.7

Theorem 4.1. *Let F be a matching, let K be a set of edges disjoint from F such that there is no $K - F$ augmenting F -alternating path. If A is an augmenting F -alternating path then there exists an edge $e \in E(A) \setminus F$ such that $OR(K \cup \{e\}, F) \supsetneq OR(K, F)$.*

Proof. Let G be the graph on V whose edge set is $K \cup F$. By the assumption that there is no $K - F$ augmenting alternating path, F is a maximal matching in G . Clearly, if the theorem is true when K is replaced by a subset L with $OR(L, F) = OR(K, F)$ then it is true also for K . Thus we may assume that condition (2) in Theorem 3.1 holds. By this theorem it follows that the pair (G, F) is an SBS. Since we may clearly assume that G is connected, it is in fact a snick-berry tree, guided by some tree T . Suppose that there exists an edge $e = uv$ of A between two distinct islands V_s and V_t ($s, t \in V(T)$). One of s, t , say s , is not a descendant of the other. By Remark 3.2 it follows that $r(t) \in OR(K \cup \{e\}, F) \setminus OR(K, F)$, which validates the theorem.

Thus we may assume that there is no edge e as above. Let V_q be the last island visited by A . Since A terminates at a non $\bigcup F$ vertex, it must leave V_q at some point, and by the above the edge of A leaving V_q must be of the form $xm(uv)$ for some vertex $x \in V_q$ and an edge uv of T . Then its next edge must be $m(uv)v$, reaching the island V_p where $v = r(p)$, contradicting the assumption that V_q is the last island visited by A . □

Given a family (namely a multiset) \mathcal{P} of F -alternating paths, an F -alternating path P is said to be \mathcal{P} -multicolored if $E(P) \setminus F$ is a partial rainbow set of the family $E(Q)$, $Q \in \mathcal{P}$.

Corollary 4.2. *If \mathcal{P} is a family of augmenting F -alternating paths and $|\mathcal{P}| > 2|F|$ then there exists an augmenting \mathcal{P} -multicolored F -alternating path.*

Proof. By Theorem 4.1 we can construct inductively sets of edges K_i , where $K_0 = \emptyset$ and $K_i = K_{i-1} \cup \{e_i\}$, $e_i \in E(P_i)$, and $OR(K_{i+1}, F) \supsetneq OR(K_i, F)$. Since there are only $2|F|$ vertices in $\bigcup F$, at some point $OR(K_i, F)$ will contain a vertex not in $\bigcup F$, meaning that there exists an augmenting $K_i - F$ -alternating path P , which by the inductive construction of the sets K_i is \mathcal{P} -multicolored. □

Finally, we derive Theorem 1.7 from Corollary 4.2. We have to show that given $3n - 2$ matchings M_i , $i \leq 3n - 2$ of size n there exists a partial rainbow matching of size n . Let F be a rainbow matching of maximal size, and let $|F| = k$. We wish to show that $k = n$. Suppose to the contrary that $k < n$. Then there are at least $2k - 1$ matchings M_i not represented in F . Each of these generates an augmenting F -alternating path P_i , and by the corollary, there is an augmenting multicolored F -alternating path P using edges from

the paths P_i . None of the colors appearing in P are used in F , and hence $F \triangle E(P)$ is a partial rainbow matching of size $k + 1$, contradicting the maximality property of k .

Remark 4.3. In [4] it was shown that in the bipartite case Corollary 4.2 only demands $|\mathcal{P}| > |F|$. In the case of general graphs Corollary 4.2 is sharp - $2|F|$ augmenting F -alternating paths do not suffice. The example showing this is essentially the same as Example 1.6. Let F be a matching $\{u_i v_i \mid i \leq k-1\} \cup \{xy\}$, let P_1, \dots, P_k all be the same path $F \cup \{xu_1\} \cup \{v_{k-1}y\} \cup \{v_i u_{i+1} \mid i \leq k-2\}$ and let P_{k+1}, \dots, P_{2k} all be equal to the same path $F \cup \{xv_1\} \cup \{u_{k-1}y\} \cup \{u_i v_{i+1} \mid i \leq k-2\}$.

Note that this is essentially Example 1.6: the paths are constructed by adding F to the matchings M_i appearing in that example.

5. A SCRAMBLED VERSION AND REFLECTIONS ABOUT THE SECRET BEHIND CONJECTURE 1.5

Should the sets M_i in Theorem 1.3 necessarily be matchings? What happens when we take $2n-1$ matchings of size n each, and scramble them, so as to obtain another system of $2n-1$ sets of edges, each of size n ? We conjecture that there still must exist a rainbow matching of size n . By König's edge coloring theorem this is equivalent to the following:

Conjecture 5.1. *Any system E_1, \dots, E_{2n-1} of sets of edges in a bipartite graph, each of size n and satisfying $\Delta(\bigcup E_i) \leq 2n-1$, has a rainbow matching of size n .*

In [3] a weaker version was proved, using topological methods:

Theorem 5.2. *Let $d \geq n^2$ and let E_1, \dots, E_d be sets of edges of size n in a bipartite graph, each of size n , and assume that $\Delta(\bigcup E_i) \leq d$. Then the sets have a rainbow matching of size n .*

Conjecture 5.1 is a special case of a very general conjecture:

Conjecture 5.3. *Let H be a 3-partite hypergraph with sides A, B, C . If $\deg(a) > \deg(u)$ for every $a \in A$ and $u \in B \cup C$ then $\nu(H) = |A|$, namely there is a matching covering A .*

Conjecture 5.1 is obtained by taking two copies, e^1, e^2 for every edge $e \in \bigcup E_i$, adding to the bipartite graph a new side containing a vertex a_i for each E_i , and defining $H = \bigcup_{i \leq 2n-1} \{a_i \cup e^1 \mid e \in E_i\} \cup \bigcup_{i \leq 2n-1} \{a_i \cup e^2 \mid e \in E_i\}$. If Conjecture 5.3 is true, then there exists a matching of size $2n-1$ in H . For either $j = 1$ or $j = 2$ this matching contains n edges e^j , which form the desired partial rainbow matching.

The parallel conjecture for general graphs is:

Conjecture 5.4. *Any system of sets of edges of size $\Delta(G) + 2$ in a general graph G has a full rainbow matching.*

In [1] this was proved when $\Delta(G) + 2$ is replaced by $\Delta(L(G)) + 2$, where $L(G)$ is the line graph of G . Example 1.6, together with the argument above linking the conjecture to Theorem 1.3, show that the requirement $\Delta(G) + 2$ is indeed necessary, namely size $\Delta(G) + 1$ of the sets does not suffice.

6. A SCRAMBLED VERSION AND REFLECTIONS ABOUT THE SECRET BEHIND CONJECTURE 1.5

Conjecture 5.1 is a special case of a daring general conjecture:

Conjecture 6.1. *In a tripartite hypergraph H on vertex classes A, B, C , if $\deg(a) \geq \deg(u) + 2$ whenever $a \in A$ and $u \in B \cup C$ then there is a matching covering the set A . If $\deg(u) > 2$ for all $u \in B \cup C$, then it is sufficient to assume that $\deg(a) \geq \deg(u) + 1$ for all $a \in A$ and $u \in B \cup C$.*

We know only one example, taken from [9, 11], necessitating the assumption that $\deg(u) > 2$ for all $u \in B \cup C$. Take three vertex disjoint copies of C_4 , say A_1, A_2, A_3 . Number the edges of A_i cyclically as a_i^j ($j = 1 \dots 4$). Let $E_1 = \{a_1^1, a_1^3, a_1^4\}$, $E_2 = \{a_2^1, a_2^4, a_2^3\}$, $E_3 = \{a_3^1, a_3^2, a_3^3\}$ and $E_4 = \{a_2^2, a_2^3, a_3^4\}$. Then $\Delta(\bigcup_{i \leq m} E_i) = 2$, $|E_i| = 3$ and there is no rainbow matching.

Conjecture 5.1 follows from Conjecture 6.1 by taking two copies, e^1, e^2 for every edge $e \in \bigcup E_i$, adding to the bipartite graph a new side containing a vertex a_i for each E_i , and defining $H = \bigcup_{i \leq 2n-1} \{a_i \cup e^1 \mid e \in E_i\} \cup \bigcup_{i \leq 2n-1} \{a_i \cup e^2 \mid e \in E_i\}$. If Conjecture 5.3 is true, then there exists a matching of size $2n - 1$ in H . For either $j = 1$ or $j = 2$ this matching contains n edges e^j , which form the desired partial rainbow matching.

The parallel conjecture for general graphs is:

Conjecture 6.2. *For any graph G , any system of sets of edges, each of size $\Delta(G) + 2$, has a full rainbow matching.*

In [1] this was proved when $\Delta(G) + 2$ is replaced by $\Delta(L(G)) + 2$, where $L(G)$ is the line graph of G . Example 1.6, together with the argument above linking the conjecture to Theorem 1.3, show that the requirement that the sets are of size $\Delta(G) + 2$ is indeed necessary, namely size $\Delta(G) + 1$ does not suffice.

Acknowledgement We are indebted to Dennis Clemens for useful remarks.

REFERENCES

- [1] R. Aharoni, N. Alon and E. Berger, Eigenvalues of $K_{1,k}$ -free graphs and the connectivity of their independence complexes, *to appear in Jour. Graph Th.*
- [2] R. Aharoni and E. Berger, Rainbow matchings and matchings in r -partite hypergraphs, *Electronic Jour. Combinatorics* (2009).
- [3] R. Aharoni, E. Berger and D. Howard, Degree conditions for rainbow matchings, *in preparation*.
- [4] R. Aharoni, D. Kotlar, R. Ziv, The extreme case of a theorem of Drisko, *submitted for publication*.
- [5] R. Aharoni, D. Kotlar, R. Ziv, A generalization of a theorem of Drisko, *submitted for publication*.
- [6] J. Barát, A. Gyárfás and J. Lehel, Rainbow matchings in bipartite multigraphs, *to appear in Periodica Mathematica Hungarica*.
- [7] A. A. Drisko, Transversals in row-Latin rectangles, *J. Combin. Theory Ser. A*, **84** (1998), 181–195.
- [8] T. S. Geisel (Dr. Seuss), *The Butter Battle Book*, Random House, NY, 1984.
- [9] G. Jin, Complete Subgraphs of r -partite Graphs, *Combinatorics, Probability and Computing* **1** (1992), : 241–250.
- [10] A. Pokrovskiy, An approximate version of a conjecture of Aharoni and Berger, <https://arxiv.org/abs/1609.06346>.
- [11] R. Yuster, Independent transversals in r -partite graphs, *Discrete Math.* **176**(1997), 255–261.

DEPARTMENT OF MATHEMATICS, TECHNION

E-mail address, Ron Aharoni: raharoni@gmail.com

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HAIFA

E-mail address, Eli Berger: eberger@haifa

DEPARTMENT OF IEOR, COLUMBIA

E-mail address, Maria Chudnovsky: mchudnov@columbia.edu

DEPARTMENT OF MATHEMATICS, COLGATE UNIVERSITY

E-mail address, David Howard: dmhoward@colgate.edu

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY

E-mail address, Paul Seymour: pds@math.princeton.edu